

# Multipath Diversity and Coding Gains of Cyclic-Prefixed Single Carrier Systems

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**Abstract**—The multipath diversity and coding gain metrics for cyclic-prefixed single-carrier (SC-CP) systems, which characterize the bit error rate (BER) at high SNR, have not been carefully studied in the literature. We first show that, unlike OFDM, the diversity and coding gains for SC-CP are *data-realization-dependent*. Then, we show that there is a signal-to-noise ratio (SNR) threshold beyond which the dominant diversity order starts deviating from the maximum diversity order to eventually reduce to one at higher SNRs. Using the averaged pairwise probability, we derive an analytical expression for this SNR threshold. The latter is shown to increase with the block length and to be unrealistically high for moderate/high block lengths. Comparisons of SC-CP with rotated constellations and zero-padded SC systems are also provided.

**Index Terms**—Multipath channels, error analysis, single carrier, OFDM, diversity.

## I. INTRODUCTION

In the case of dispersive channels, serial transmissions often require complex equalizers. Further, linear zero-forcing equalizers do not exist for such transmissions. To mitigate these problems, block transmission techniques with a guard interval between two consecutive blocks were introduced. When the duration of the guard interval is at least equal to the delay spread of the channel, interblock interference is avoided and symbol detection can be performed on a per-block basis. Two options are commonly used when designing the guard interval: a cyclic prefix (CP) or zero-padding (ZP). A CP turns the convolution of the transmitted data symbols (i.e., outputs of the symbol mapping stage) and the channel into a circular convolution, which simplifies channel equalization. Orthogonal Frequency Division Multiplexing (OFDM) is an example of a CP system, where the data symbols are first linearly precoded by an inverse DFT matrix. If the data symbols are not linearly precoded, the system is commonly referred to as cyclic-prefixed single-carrier (SC-CP) system. In ZP single-carrier (SC-ZP) systems, a set of zero symbols are inserted between consecutive blocks. It was shown in [1] that among the class of linearly precoded block transmissions over multipath Rayleigh channels, SC-ZP systems have maximum diversity and coding gains when maximum likelihood (ML) decoding is employed. Comparisons between OFDM and SC-CP or SC-ZP have been carried out in many publications e.g., [1], [2] and references therein. It was mentioned in [3] that the multipath diversity for SC-CP is of order one, which would suggest that for *reasonably* high signal-to-noise ratios (SNR),

the slope of the bit error rate curve is equal to one. However, simulations results show that this is not the case. In Figure 1, the BER for the maximum likelihood detector for a SC-CP system with a block length  $N = 8$  and channel length  $L = 4$  is equal to those of SC-ZP and rotated-constellation SC-CP schemes, which were shown to capture maximum multipath diversity, for signal-to-noise ratios (SNR) smaller than 13dB, but beyond this value the slope of the BER curve starts to deviate from  $L$  to eventually reduce to one at higher SNRs. This SNR threshold depends on  $N$  and  $L$  and to a lesser extent on the channel statistics. This threshold becomes impractically high when  $N$  becomes moderate/large, which is often the case in practice. In this paper, we explain why this phenomenon occurs and we also give an analytical expression to predict the SNR threshold.

**Notation:** Boldface small (resp. capital) letters denote vectors (resp. matrices.) The  $(N \times N)$  DFT matrix is defined as  $\mathbf{F} = 1/\sqrt{N}\{\exp(-j2\pi nm/N)\}_{n,m=0}^{N-1}$ .  $\mathbf{D}_b$  will denote the diagonal matrix whose diagonal is  $\mathbf{b}$ . Superscripts  $*$  and  $T$  denote Hermitian and transpose operators. The  $L_2$  norm, trace, rank and statistical expectation are denoted by  $\|\cdot\|$ ,  $\text{trace}\{\cdot\}$ ,  $\mathcal{R}\cdot$  and  $\mathbb{E}[\cdot]$ . The probability of event  $A$  is denoted by  $\Pr(A)$ .

## II. SIGNAL MODEL FOR CYCLIC-PREFIXED SYSTEMS

Consider a block transmission system where a cyclic prefix (CP), whose duration exceeds the delay spread of the multipath channel, is inserted between consecutive blocks of length  $N$  in order to avoid interblock interference. Assume that the frequency-selective channel is time-invariant over a frame of several blocks. Let  $\mathbf{h} = [h_0 \cdots h_{L-1}]^T$  denote the baud-rate sampled discrete-time channel impulse response (CIR) during the frame. After removing the CP, the  $i$ th block of the received frame can be modeled as

$$\mathbf{x}_i = \mathbf{H}\mathbf{u}_i + \mathbf{v}_i, \quad (1)$$

where  $\mathbf{u}_i$  is the  $i$ th transmitted block, and  $\mathbf{H}$  is an  $(N \times N)$  circulant matrix with first column  $[h_0 \ h_1 \ \cdots \ h_{L-1} \ 0 \ \cdots \ 0]^T$  and  $\mathbf{v}_i$  is an AWGN vector with variance  $\sigma^2$ . Since circular matrices can be diagonalized using DFT matrices, the above signal model can also be rewritten as

$$\mathbf{x}_i = \mathbf{F}^* \mathbf{D}_{\tilde{\mathbf{h}}} \mathbf{F} \mathbf{u}_i + \mathbf{v}_i \quad (2)$$

where  $\tilde{\mathbf{h}} = [\tilde{h}_0 \cdots \tilde{h}_{N-1}]^T$  with  $\tilde{h}_n$  being the frequency response of the channel at the normalized frequency  $n/N$ , i.e.,  $\tilde{h}_n = \sum_{\ell=0}^{L-1} h_\ell \exp(-j2\pi n\ell/N)$ .

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We consider linear precoding where  $\mathbf{u}_i = \Phi \mathbf{s}_i$  with  $\Phi$  being an  $(N \times N)$  matrix and  $\mathbf{s}_i$  the  $i$ th data block. The elements of  $\mathbf{s}_i$  are assumed to have unit variance, without loss of generality. The choice of matrix  $\Phi$  greatly affects the bit error rate performance, particularly for uncoded systems. In what follows, we will focus on three CP-based systems: i) SC-CP where  $\Phi = \mathbf{I}$ , ii) OFDM where  $\Phi = \mathbf{F}^*$ , and iii) rotated constellations (RC-) SC-CP where  $\Phi = \text{diag}(1, e^{j\alpha}, \dots, e^{j(N-1)\alpha})$ . We only consider uncoded systems, i.e., systems without inner coding. We make the following assumptions:

(A1) the data symbols, i.e., elements of  $\mathbf{s}_i$ , are i.i.d. random variables drawn from a finite-alphabet, e.g., PSK, PAM or QAM constellations;

(A2) the channel vector is zero-mean, complex Gaussian with full rank correlation matrix  $\mathbf{R}_h := \mathbb{E}[\mathbf{h}\mathbf{h}^*]$ .

### III. MULTIPATH DIVERSITY AND CODING GAINS

In this section, we also make the following assumption:

(A3) maximum-likelihood (ML) detection is performed with perfect channel state information (CSI) available at the receiver, but not at the transmitter.

Exact bit-error rate (BER) performance analysis is difficult if not impossible. Hence, we resort to the pairwise error probability (PEP) analysis which provides a good approximation of the BER at high SNR; see e.g., [4],[5],[6],[7]. Since the PEP analysis is performed on a per-block basis, we drop the block index  $i$  for notational simplicity.

We define the pairwise error event  $\{s \rightarrow s'\}$  with  $s' \neq s$  as the event that the output of the decoder is  $s'$  when  $s$  is transmitted. Let  $P\{s \rightarrow s'\}$  denote the corresponding PEP which can, using the Chernoff bound, be upper-bounded as follows [4],[5],[6],[7]

$$\Pr(s \rightarrow s' | \mathbf{h}) \leq \exp \left[ -\frac{d^2(\mathbf{x}', \mathbf{x})}{4\sigma^2} \right] \quad (3)$$

where  $\mathbf{x} := \mathbf{F}^* \mathbf{D}_h \mathbf{F} \Phi \mathbf{s}$ ,  $\mathbf{x}' := \mathbf{F}^* \mathbf{D}_h \mathbf{F} \Phi \mathbf{s}'$  and  $d(\mathbf{x}', \mathbf{x}) = \|\mathbf{x}' - \mathbf{x}\|$  is the Euclidean distance between  $\mathbf{x}'$  and  $\mathbf{x}$ . Since  $\mathbf{F}$  is unitary, we have that  $d(\mathbf{x}', \mathbf{x}) = \|\mathbf{D}_h \mathbf{F} \mathbf{e}\|$  where  $\mathbf{e} := \mathbf{F} \Phi (\mathbf{s}' - \mathbf{s})$ . The average PEP over all possible channel realizations can, under assumption (A2), be expressed as [6][7]

$$\Pr(s \rightarrow s') \leq \prod_{l=0}^{\mathcal{R}(\mathbf{C}_e)-1} \left( 1 + \frac{\lambda_l}{4\sigma^2} \right)^{-1} \quad (4)$$

where  $\lambda_l$ ,  $l = 0, \dots, \mathcal{R}(\mathbf{C}_e) - 1$  are nonzero eigenvalues of  $\mathbf{C}_e$  which is defined as

$$\mathbf{C}_e = N \mathbf{R}_h^{1/2} \mathbf{F}_L^H \mathbf{D}_e^H \mathbf{D}_e \mathbf{F}_L \mathbf{R}_h^{1/2} \quad (5)$$

where  $\mathbf{D}_e = \text{diag}(\mathbf{e})$  and  $\mathbf{F}_L$  is the leading  $(N \times L)$  submatrix of  $\mathbf{F}$ . When the SNR is high, we obtain

$$\Pr(s \rightarrow s') \leq \left( \frac{G_c(\mathbf{s}, \mathbf{e})}{4\sigma^2} \right)^{-G_d(\mathbf{s}, \mathbf{e})} \quad (6)$$

where  $G_d(\mathbf{s}, \mathbf{e}) = \mathcal{R}(\mathbf{C}_e)$  and  $G_c(\mathbf{s}, \mathbf{e}) = \left( \prod_{l=0}^{\mathcal{R}(\mathbf{C}_e)-1} \lambda_l \right)^{1/\mathcal{R}(\mathbf{C}_e)}$ . The argument  $\mathbf{s}$  is introduced to stress the fact that the possible values of  $\mathbf{e}$  depend on the transmitted vector  $\mathbf{s}$  due to the finite alphabet property. This is unlike in [7] where  $G_c$  and  $G_d$  were only functions of  $\mathbf{e}$ .

Under (A2), we have that

$$G_d(\mathbf{s}, \mathbf{e}) = \min(L, K)$$

where  $K$  is the number of nonzero entries of  $\mathbf{e}$ . The diversity gain corresponding to a realization of  $\mathbf{s}$  is given by

$$G_d(\mathbf{s}) = \min_{\mathbf{e} \neq 0} G_d(\mathbf{s}, \mathbf{e})$$

The coding gain conditioned on  $\mathbf{s}$  is defined as

$$G_c(\mathbf{s}) = \min_{\mathbf{e} \neq 0} G_c(\mathbf{s}, \mathbf{e})$$

The PEP is thus upper bounded as follows:

$$\Pr(s \rightarrow s') \leq \left( \frac{G_c(\mathbf{s})}{4\sigma^2} \right)^{-G_d(\mathbf{s})} \quad (7)$$

Note that the diversity and coding gains in [7] were implicitly defined as  $G_d = \min_{\mathbf{s}} \min_{\mathbf{e} \neq 0} G_d(\mathbf{s}, \mathbf{e})$  and  $G_c = \min_{\mathbf{s}} \min_{\mathbf{e} \neq 0} G_c(\mathbf{s}, \mathbf{e})$ . Such measures of performance will be shown later to be very pessimistic for CP-SC as far as average performance is concerned.

#### A. Maximum diversity and coding gains

The maximum diversity and coding gains for CP linearly precoded systems are similar to the ones derived in [7] for CP-OFDM. Indeed, by replacing  $\Phi \mathbf{F}^H$  in Fig.1 of [7] by our matrix  $\Phi$ , we obtain the same signal model as in [7]. Thus, under (A2), the maximum diversity gain is  $G_{d,\max} = L$  and this is achieved when  $K \geq L$ , i.e.,  $\mathcal{R}(\mathbf{C}_e) \geq L$ . Assuming maximum diversity, the maximum coding gain is given by [7]

$$G_{c,\max} = [\det \mathbf{R}_h]^{1/L} \Delta_{\min}^2 \quad (8)$$

where  $\Delta_{\min}$  denotes the minimum Euclidean distance among constellation points in  $\mathcal{A}$ .

#### B. OFDM

Here,  $\mathbf{e} = \mathbf{s}' - \mathbf{s}$  and  $G_d(\mathbf{s}, \mathbf{e})$  is minimized when  $\mathbf{e}$  has one non-zero element only, i.e.,  $K = 1$ . Such an error vector is possible for any data vector  $\mathbf{s}$ . Hence, the diversity gain is data-realization-independent and it is always equal to one, i.e.,  $G_d(\mathbf{s}) = G_d = 1, \forall \mathbf{s}$ . Using eq. (5), the corresponding coding gain is, after some algebra, found to be

$$G_c^{\text{OFDM}} = \Delta_{\min}^2 \sigma_{h,\min}^2 \quad (9)$$

where  $\sigma_{h,\min}^2 = \min_{m=0 \dots N-1} \sigma_h^2(m)$  and  $\sigma_h^2(m) = \mathbb{E}[|\tilde{h}_m|^2]$  with  $\tilde{h}_m$  being the frequency response of the channel at frequency bin  $m/N$ . This averaged channel frequency response is flat only if the channel taps are uncorrelated, in which case  $\sigma_h^2(m) = \text{Tr}\{\mathbf{R}_h\}, \forall m$ . Therefore, the PEP for uncoded OFDM satisfies

$$\Pr(s \rightarrow s') \leq \frac{1}{\text{SNR}} \frac{4 \text{Tr}\{\mathbf{R}_h\}}{\Delta_{\min}^2 \sigma_{h,\min}^2} \quad (10)$$

Further, the PEP for each subcarrier satisfies

$$P(s_m \rightarrow s'_m) \leq \frac{1}{\text{SNR}_m} \frac{4}{\Delta_{\min}^2} \quad (11)$$

where  $\text{SNR}_m = \sigma_h^2(m)/\sigma^2$  is the average SNR at the  $m$ th carrier. For uncorrelated-tap channels, the SNR across the

subcarriers are the same and thus the individual PEPs become equal to each other. A tighter bound can be obtained because the ML detections of the data symbols are decoupled in OFDM systems.

### C. SC-CP Systems

In this case,  $\mathbf{e} = \mathbf{F}(\mathbf{s}' - \mathbf{s})$ . The multipath diversity gain for SC-CP is one *only* for the realizations of  $\mathbf{s}$  for which there exists an error vector  $\mathbf{e}$  which contains one non-zero element only. Under assumption (A1), the probability of this occurring is an exponentially decreasing function of  $N$ . For example for BPSK constellations  $G_d(\mathbf{s}) = 1$  only for 4 sequences out of  $2^N$  if  $N$  is even; these are  $\mathbf{s} = \pm[1, 1, \dots, 1]^T$ ,  $\mathbf{s} = \pm[1, -1, 1, -1, \dots, -1]^T$ . More generally,  $P_i := \Pr(G_d = i)$ , which is related to the  $L_0$  norm of  $\mathbf{e}$  and can be determined via an exhaustive search over all possible vectors  $\mathbf{s}$ . For  $L = 4$  and if  $N$  is an integer multiple of 4, we have that  $P_1 = P_2 = 2^{2-N}$ ,  $P_3 = 0$  and  $P_4 = 1 - 2^{3-2N}$ . The following Lemma will be useful to derive the SNR threshold in the Theorem below.

**Lemma** *For BPSK constellations, when  $N$  is a power of two and  $L \leq N/2$ , we have that  $P_1 = 2^{2-N}$ , and  $P_L = 1 - 2^{L-N}$  if  $L-1$  is a power of two and  $P_L = 1 - 2^{L-1-N}$  otherwise.*

For  $N \geq 16$ , the probability of achieving maximum diversity gain is almost one. This is in contrast with uncoded OFDM where  $G_d(\mathbf{s}) = 1$  for all realization of  $\mathbf{s}$  and for all values of  $N$ . This explains why SC-CP outperforms uncoded OFDM.

However the probability of wrong detection of a diversity-one data block is higher than that of data blocks which capture higher multipath diversity. This implies that although the probability of diversity one is low, it may significantly affect the average (over both the channel and the data) detection performance. This will depend on how high the SNR is and on the data block length.

The averaged (over the data) PEP (APEP) satisfies

$$\text{APEP} := \mathbb{E}_{\mathbf{s}}[\Pr(\mathbf{s} \rightarrow \mathbf{s}')] \leq \frac{1}{|\mathcal{A}|^N} \sum_{\mathbf{s} \in \mathcal{A}^N} \left( \frac{G_c(\mathbf{s})}{4\sigma^2} \right)^{-G_d(\mathbf{s})}$$

where  $|\mathcal{A}|$  denotes the cardinality of  $\mathcal{A}$ . To gain better insight into the multipath diversity gain issue, we further loosen the upper bound on the APEP as follows

$$\text{APEP} \leq \sum_{i=1}^L P_i \left( \frac{G_c^{(i)}}{4\sigma^2} \right)^{-i} \quad (12)$$

where  $G_c^{(i)}$  is defined as the minimum coding gain associated with diversity order  $i$ , i.e.,

$$G_c^{(i)} := \min_{\mathbf{s}; G_d(\mathbf{s})=i} G_c(\mathbf{s})$$

Moreover for high SNR, we can neglect the terms related to diversity orders  $i = 2, \dots, L-1$  compared to the diversity-one term. Indeed, although  $P_{i+1} > P_i$  for  $i < L-2$ , this increase is not high enough compared to the decrease due to  $\sigma^{-2}$ ; the corresponding coding gains are close to each other. Since, for typical values of  $N$  and  $L \ll N$ ,  $P_i \ll 1$  when  $i < L$ ,

and  $P_L$  is close to one, the maximum diversity-related term, although proportional to  $\sigma^{2L}$ , may dominate the diversity-one-related term. Therefore, we approximate the upper bound on the APEP by the terms of the RHS of (12) related to diversity one and maximum diversity,  $L$ , and neglect the other terms.

The coding gain associated with maximum diversity order is given in eq. (8). In the case of BPSK, the coding gain when  $G_d(\mathbf{s}) = 1$  can be shown to be given by

$$G_c^{(1)} = N \Delta_{\min}^2 \min(\sigma_h^2(0), \sigma_h^2(N/2)) \quad (13)$$

The reason why only frequency bins  $m = 0, N/2$  are relevant in the coding gain expression is due to the fact that the possible positions of the only nonzero element of vector  $\mathbf{e}$  are  $m = 0, N/2$ , since  $\mathbf{F}^* \mathbf{e}$  must be one of the following sequences  $\pm[2, 2, \dots, 2]^T$ ,  $\pm[2, -2, 2, -2, \dots, -2]^T$ , which are obtained from  $\mathbf{s} - \mathbf{s}'$  where  $\mathbf{s}$  and  $\mathbf{s}'$  are 'diversity-one' BPSK sequences and  $\mathbf{s} \neq \mathbf{s}'$ . For higher size constellations, the coding gain may depend on other frequency bins. So, we approximate the coding gain by a lower bound as follows

$$G_c^{(1)} \approx N \Delta_{\min}^2 \sigma_{h,\min}^2 \quad (14)$$

which is valid for any constellation size. As mentioned previously, if the channel taps are uncorrelated, the  $\sigma_h^2(m)$ 's are all equal to  $\text{Tr}\{\mathbf{R}_h\}$ , and thus the approximation in eq. (14) coincides with the coding gain in eq. (13). Therefore, an approximate upper bound on the APEP is given in the following theorem.

**Theorem** *Under assumptions (A1)-(A3), the APEP at high SNR approximately satisfies*

$$\text{APEP} \leq P_L \left( \frac{[\det \mathbf{R}_h]^{\frac{1}{L}} \Delta_{\min}^2}{4\sigma^2} \right)^{-L} + P_1 \left( \frac{N \Delta_{\min}^2 \sigma_{h,\min}^2}{4\sigma^2} \right)^{-1}$$

where  $P_i$  is the probability that the captured multipath diversity order is  $i$ , which depends on  $N$ ,  $L$  and constellation type and size.

In order to find the SNR threshold beyond which the BER performance of SC-CP ceases to behave like that of maximum-diversity and maximum coding gain schemes, such as ZP and RC-SC-CP, we propose to compare the terms in the above upper bound on the APEP. The SNR is defined as  $\text{SNR} = \text{Tr}\{\mathbf{R}_h\}/\sigma^2$  since we assume  $\mathbb{E}[|s_n|^2] = 1$ . We will say that SC-CP is optimal among linear precoding schemes if the maximum-diversity-related term in the above upper bound on the APEP is much higher, say 10 times higher, than the term associated with diversity order one.

$$P_L \left( \frac{G_c^{(L)}}{4\sigma^2} \right)^{-L} \geq 10 P_1 \left( \frac{G_c^{(1)}}{4\sigma^2} \right)^{-1} \quad (15)$$

Define the normalized channel matrix as  $\bar{\mathbf{R}}_h := \mathbf{R}_h / \text{Tr}\{\mathbf{R}_h\}$  and define  $\bar{\sigma}_h^2(m) := \sigma_h^2(m) / \text{Tr}\{\mathbf{R}_h\}$ .

**Corollary** *Under assumptions (A1)-(A3), SC-CP is an optimal linear precoding scheme if*

$$\text{SNR} \leq \frac{4}{\Delta_{\min}^2} \left( \frac{N P_L}{10 P_1} \right)^{\frac{1}{L-1}} \left( \frac{\bar{\sigma}_{h,\min}^2}{\det \bar{\mathbf{R}}_h} \right)^{\frac{1}{L-1}} \quad (16)$$

If  $L = N/4$  and  $\mathbf{R}_h = \text{diag}(\exp(-0.2\ell), \ell = 0, \dots, L-1)$ , the SNR threshold is equal to 23dB, 25dB, 28dB, 34dB and 48dB when  $N$  is equal to 16, 32, 64, 128 and 256, respectively. Hence, in practical wireless communication systems where  $N \geq 64$ , the threshold on the average SNR may be too optimistic in wireless systems.

Similarly, the SNR threshold beyond which errors related to diversity order one dominate the APEP can be obtained from the RHS of eq. (16) after replacing 10 by 0.1.

Although the above analysis assumes a high SNR regime, simulations show that SC-CP behaves like SC-ZP and RC-SC-CP for all SNRs below the threshold given above.

#### D. Rotated-Constellations SC-CP

The idea of rotating the constellations to capture maximum diversity was proposed in the context of MIMO channels. Inspired by this, the RC-SC-CP scheme was proposed in [3], [1]. For SC-CP systems, such a rotation insures that the  $L_0$  norm of the error vector  $\mathbf{e}$  is at least equal to  $L$ , which implies full multipath diversity. If  $N$  is a power of two, and for PSK and QAM constellations, the following rotation is sufficient to obtain this property:  $\Phi = \text{diag}(1, e^{j\alpha}, \dots, e^{j(N-1)\alpha})$  with  $\alpha = \pi/2$ . The coding gain is given in eq. (8). Although such rotation and the corresponding de-rotation at the receiver are computationally non-demanding and achieve maximum diversity and coding gain, our analysis show that for practical values of  $N$  and  $L$ , these operations are not useful for practical values of the SNR.

#### IV. SIMULATIONS AND DISCUSSIONS

We consider OFDM, SC-CP, SC-ZP and RC-SC-CP systems, with  $\mathbf{R}_h = \text{diag}(\exp(-0.2\ell), \ell = 0, \dots, L-1)$ ,  $L = 4$ ,  $N = 8$  and BPSK modulation. Figure 1 displays the BER results for the four systems. ML (implemented using sphere decoding), zero-forcing (ZF) and minimum-mean square error (MMSE) detection schemes are considered. The SNR threshold predicted by the above Corollary is 13.8dB. This is verified by the results in Figure 1, where the BER of ML detection for SC-CP coincides with those of SC-ZP and RC-SC-CP for SNRs up to about the predicted threshold. Beyond this SNR threshold, the BER for SC-CP diverges from those of RC-SC-CP and SC-ZP. We note that the slope of the BER for SNRs below but close to 13.8dB is not exactly  $L = 4$  is due to the fact that the true Chernoff bound for diversity order  $i$  is proportional to  $(1 + \alpha \text{SNR})^{-i}$ ,  $\alpha$  is a constant, not to  $\text{SNR}^{-i}$  unless SNR is very high.

Figure 1 also shows that although the multipath diversity order of SC-ZP with linear detection is not maximum, it is much larger than one. As with ML detection, Figure 1 shows that MMSE detection performance of SC-CP is close to that of SC-ZP for SNR values below the threshold given in the Corollary. Beyond this threshold, the BER of SC-CP starts to significantly deviate from that of SC-ZP to eventually exhibit a unity diversity order at much higher SNRs. Thus, BER performance of SC-CP is close to that of SC-ZP, for both ML and MMSE detectors, for typical values of  $N$ ,  $L$  and SNR values. Note also that MMSE detection for SC-ZP requires

inversion of an  $((N + L - 1) \times N)$  matrix, whereas that for SC-CP is significantly less complex since it can be performed using an FFT, one tap equalizations and an IFFT. When battery life is an issue, SC-ZP may be preferred to SC-CP for signal transmission since power used to transmit a CP in SC-CP is saved in SC-ZP. Thus, in mobile communications, a hybrid system where SC-CP is used for the downlink and SC-ZP is used for the uplink seems appropriate since it addresses both computational complexity and battery life at the mobile unit.

#### V. CONCLUSIONS

Multipath diversity and coding gains for SC-CP transmission over Rayleigh channels were analyzed using a probabilistic approach. Using an upper bound on the average pairwise error probability of maximum likelihood detection and simulations, it was argued that SC-CP behaves like a maximum diversity and coding gain scheme (e.g. SC-ZP) for SNRs smaller than a threshold which becomes unrealistically high for typical block lengths. This threshold behavior was observed with the MMSE detector as well.

#### REFERENCES

- [1] Z. Wang, X. Ma and G. B. Giannakis, "OFDM or single-carrier block transmissions?," *IEEE Trans. Communications* 52(3), 380-394, March 2004.
- [2] F. Pancaldi *et al.*, "Single-carrier frequency domain equalization," *IEEE Signal Processing Magazine*, 25(5), 37-58, September 2008.
- [3] S. Zhou and G. B. Giannakis, "Single-Carrier Space-Time Block Coded Transmissions over Frequency-Selective Fading Channels," *IEEE Trans. Information Theory* 49(1), 164-179, January 2003.
- [4] A.J. Viterbi and J.K. Omura, *Principles of digital communication and coding*, McGraw Hill, 1979.
- [5] J. Proakis, *Digital Communications*, 4th edition, McGraw Hill, 2000.
- [6] A.F. Naguib *et al.*, "A space-time coding modem for high-data-rate wireless communications", *IEEE Journal on Selected Areas in Communications*, 16(8), 1459-1478, Oct. 1998.
- [7] Z. Liu, Y. Xin and G. B. Giannakis, "Linear constellation precoding for OFDM with multipath diversity and coding gains," *IEEE Trans. Communications* 51(3), 416-27, March 2003.

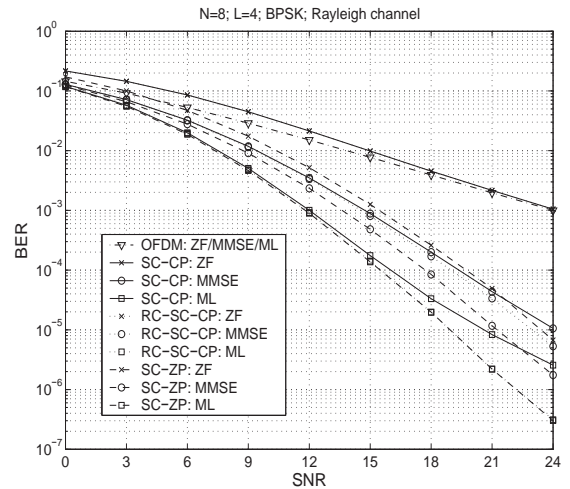


Fig. 1: Bit error rate versus SNR,  $N = 8$ .